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COVERING 3-COLOURED RANDOM GRAPHS WITH MONOCHROMATIC TREES

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Abstract. We investigate the problem of determining how many monochromatic trees are necessary to cover the vertices of an edge-coloured random graph. More precisely, we show that for $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$ in any 3-colouring of the random graph $G(n,p)$ we can find 3 monochromatic trees such that their union covers all vertices. This improves, for three colours, a result of Bucić, Korándi and Sudakov [*Covering random graphs by monochromatic trees and Helly-type results for hypergraphs*, `arXiv:1902.05055`]

1. Introduction

The investigation of questions concerning covering graphs with monochromatic components started with Gerencsér and Gyárfás [**5**], who proved that in any 2-colouring of the edges of K_n there are two monochromatic paths that cover the vertex set of K_n. Pokrovskiy [**10**] proved a similar result for three colours, namely, he showed that the vertices of any 3-edge-coloured K_n can be covered with three monochromatic paths. However, for $r \geq 4$ colours it is not known if it is possible to cover the vertices an r-edge-coloured K_n with r monochromatic paths. In the last decades, many researchers have investigated the problem of covering (or partitioning) coloured graphs with monochromatic components (see, e.g., [**7**] for a survey).

Initiating the study of covering random graphs by monochromatic components, Bal and DeBiasio [**2**] posed a conjecture concerning the threshold for the following property: every r-colouring of the edges of G yields r monochromatic trees that

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cover (or partition) the vertex set of G. Given a graph $G = (V, E)$, let $\mathrm{tc}_r(G)$ denote the minimum number k such that in any r-colouring of E, there are k monochromatic trees $T_1, \ldots, T_k$ such that

$$V(T_1) \cup \cdots \cup V(T_k) = V\,,$$

i.e., V can be covered by at most k sets each containing a spanning monochromatic tree.

Bal and DeBiasio [**2**] proved the following result.

Theorem 1. *Let $G = G(n,p)$. For any integer $r \geq 2$, there exists $C \geq r$ such that if $p \geq (\frac{C \ln n}{n})^{1/(r+1)}$, then a.a.s. $\mathrm{tc}_r(G) \leq r^2$.*

They also made the following conjecture.

Conjecture 2. Let $G = G(n,p)$. For any integer $r \geq 2$ and $\varepsilon > 0$, if $p \geq (1+\varepsilon)\left(\frac{r \ln n}{n}\right)^{1/r}$, then a.a.s. we have $\mathrm{tc}_r(G) \leq r$.

A stronger version of Conjecture 2 (obtaining a partition instead of a covering) for two colours was confirmed by the first and third authors together with Schacht [**9**], but however the third author, Ebsen and Schnitzer showed that this conjecture does not hold for more than two colours (see [**9**, Proposition 4.1]). More precisely, they showed that for $r \geq 3$, we have $\mathrm{tc}_r(G(n,p)) \geq r+1$, a.a.s., for $p \ll (\ln n/n)^{1/(r+1)}$. Recently, Bucić, Korándi and Sudakov [**3**] showed that the right threshold for the property $\mathrm{tc}_r(G) \leq r$ is far from being known. More precisely they proved the following result.

Theorem 3 (Bucić, Korándi and Sudakov [**3**, Theorem 1.1])**.** *For any positive integer r there are constants c and C such that, for $G = G(n,p)$,*

(i) *if $p < \left(\frac{c \ln n}{n}\right)^{\sqrt{r}/2^{r-2}}$, then a.a.s. $\mathrm{tc}_r(G) \geq r+1$, and*
(ii) *if $p > \left(\frac{C \ln n}{n}\right)^{1/2^r}$, then a.a.s. $\mathrm{tc}_r(G) \leq r$.*

They also proved the following result that provides bounds for $tc_r(G(n,p))$ in some intermediate range of p.

Theorem 4 (Bucić, Korándi and Sudakov [**3**, Theorem 1.4])**.** *For any integers $k > r \geq 2$ there are constants c and C such that, for $G = G(n,p)$, if $\left(\frac{C \ln n}{n}\right)^{1/k} < p < \left(\frac{c \ln n}{n}\right)^{1/(k+1)}$, then a.a.s. $\frac{r^2}{20 \ln k} \leq \mathrm{tc}_r(G) \leq \frac{16 r^2 \ln r}{\ln k}$.*

Theorem 3, in particular, implies that we have $\mathrm{tc}_3(G(n,p)) \leq 3$ a.a.s., for $p \gg \left(\ln n/n\right)^{1/8}$. On the other hand, Theorem 4 implies that $\mathrm{tc}_3(G(n,p)) \leq 88$ a.a.s., for $(\ln n)/n)^{1/6} \ll p \ll (\ln n/n)^{1/7}$. Our result improves those bounds.

Theorem 5. *Let $G = G(n,p)$. If $p = p(n) \gg \left(\frac{\ln n}{n}\right)^{1/6}$, then a.a.s. $\mathrm{tc}_3(G) \leq 3$.*

From the example described in [**9**], we have that a.a.s. $\mathrm{tc}_3(G(n,p)) \geq 4$, for $p \ll \left(\frac{\ln n}{n}\right)^{1/4}$. It would be very interesting to describe the behaviour of $\mathrm{tc}_r(G)$ when $\left(\frac{\ln n}{n}\right)^{1/4} \ll p \ll \left(\frac{\ln n}{n}\right)^{1/6}$.

In the proof of Theorem 5, after being given a 3-colouring to the edges of $G = G(n,p)$, we create an auxiliary graph F as follows: F has the same vertex set of G and there is an edge between two vertices u and v in F if and only if there is a monochromatic path between u and v in G. We colour the edges of F by giving any edge uv the colour of the monochromatic path between u and v in G. Then, we analyse three cases, depending on the value of $\alpha(F)$, and show that in any case we can cover $V(G)$ with at most three monochromatic trees. See Section 2 for an elaboration of this sketch.

2. Proof overview

In this section we will give an overview of the proof of Theorem 5. Let $G = G(n,p)$, with $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$, and let $c\colon E(G) \mapsto \{\text{red}, \text{green}, \text{blue}\}$ be any 3-edge-colouring of G. We consider an auxiliary graph F, with $V(F) = V(G)$ and $ij \in E(F)$ if and only if there is a monochromatic path in G (with respect to the colouring c). Then we define a 3-edge-colouring c' of F with $c'(ij)$ being the color of any monochromatic path in G connecting i to j. Note that any covering of F with monochromatic trees (with respect to the colouring c') corresponds to a covering of G with monochromatic trees (with respect to the colouring c) with the same number of trees.

We now consider different cases depending on the value of $\alpha(F)$. If $\alpha(F) = 1$, then F is a complete 3-coloured graph and therefore, by a result of Erdős, Gyárfás and Pyber [**4**], there exists a partition of $V(F)$ into 2 monochromatic trees and thus a covering of G with 2 monochromatic trees. The proof now is divided into two cases: (i) $\alpha(F) = 2$ and (ii) $\alpha(F) \geq 3$.

Let us consider the case (ii) first. In this case, there exist three vertices $r, b, g \in V(G)$ that pairwise do not have any monochromatic path connecting them. They a.a.s. have a common neighbourhood of size at least $np^3/2$. Let J be the largest subset of this common neighbourhood such that for each $i \in \{r, b, g\}$, the edges from i to J are all coloured with one colour. Then $|J| \geq np^3/12$ and since there are no monochromatic paths between any two of r, b, g, we may assume that all edges from r to J are red, all from b to J are blue and those from g to J are green. Now we notice that all vertices that have a neighbour in J are covered by the union of the spanning trees of the red component of r, the blue component of b and the green component of g.

So we would be done if every vertex has a neighbour in J. If this is not the case, then let $Y = V \setminus \mathrm{N}(J)$ be the set of those vertices that have no neighbour in J. By the value of p, we get that a.a.s. every vertex $y \in Y$ will have many common neighbours with r, g and b that are also neighbour of some vertex in J. With some careful analyse on the possible colouring of the edges of those common neighbours, we are able to show that for some $i \in \{r, g, b\}$ (let us say, w.l.o.g., $i = r$), every vertex $y \in Y$ can be connected to r by a monochromatic path in colour red or either to g or b by a monochromatic path in the colour blue or green, respectively.

This already gives us that all vertices in G can be covered by 5 monochromatic trees, since all the vertices in $N(J)$ lie in the red component of r, or the green component of g, or in the blue component of b and every vertex in $V \smallsetminus N(J)$ lie in the blue component of g or in the green component of b. By analysing the colours of edges to the common neighbourhood of carefully chosen vertices, we are able to show that actually three of those five trees already cover all the vertices of G.

Now, we only have case (i) left. Before diving into the idea of the proof for the case (i), let us do a small regression. Let $\mathcal{H}$ be a hypergraph. We say that a set A of hypervertices of $\mathcal{H}$ is a *vertex cover* if every hyperedge of $\mathcal{H}$ has at least one element of A. The *covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the smallest size of a vertex cover in $\mathcal{H}$. A *matching* in $\mathcal{H}$ is a collection of disjoint hyperedges in $\mathcal{H}$. The *matching number* of $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the largest size of a matching in $\mathcal{H}$. A conjecture by Ryser [**8**] states that for every r-uniform r-partite hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$. Aharoni [**1**] proved that Ryser's conjecture holds for $r = 3$, but for larger r, it is still open.

Given a graph G and an r-edge-colouring of G, let us consider a hypergraph $\mathcal{H}$ defined as follows (such a construction is due to Gyárfás [**6**]). The hypervertices of $\mathcal{H}$ are the monochromatic components of F and r hypervertices form a hyperedge if the corresponding r monochromatic components have a non empty intersection (in particular they must be of different colours). Hence $\mathcal{H}$ is a r-uniform r-partite hypergraph. Now observe that $\mathrm{tc}_r(G) \leq \tau(\mathcal{H})$, for if A is vertex cover of $\mathcal{H}$, then the monochromatic components associated to the hypervertices in A cover all the vertices of G. In fact, if $v \in V(G)$ is not covered by those monochromatic components associated to the vertex cover A, then the monochromatic components of each colour containing v form a hyperedge of $\mathcal{H}$ which does not intersect A, contradicting the fact that A is vertex cover of $\mathcal{H}$. Further, notice that $\nu(\mathcal{H}) \leq \alpha(G)$ because for each matching $E_1, \ldots, E_k$ in $\mathcal{H}$ we can choose distinct vertices $v_1, \ldots, v_k$, each v_i belonging to the intersection of the r monochromatic components associated to E_i. Then if we had $k > \alpha(G)$, two vertices among $v_1, \ldots, v_k$ would be adjacent and would therefore share one monochromatic component. But that would mean that their corresponding hyperedges intersect. The two observations above give us that Ryser's conjecture implies that $\mathrm{tc}_r(G) \leq (r-1)\alpha(G)$.

Now, back to our proof, we are considering case (i) where we have $\alpha(F) = 2$. The observations in the previous paragraph already imply that $\mathrm{tc}_3(F) \leq 4$. But we want to prove that $\mathrm{tc}_3(F) \leq 3$. To this aim, we make use of Gyárfás' construction more carefully, reducing the situation to few possible settings of components covering all vertices. In each of those, we can again analyse the possible colourings of edges to the common neighbourhood of specific vertices, inferring that indeed 3 monochromatic components cover all vertices.

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